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Grids and Their Minors

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We define graphs Y_n such that none of these Y_n has a minor isomorphic to K_8 and for every number k and surface S not all of these graphs are in the completion of k-vertex extensions of graphs embeddable in S. This disproves a corresponding conjecture of N. Robertson and P. D. Seymour [*in* "Progress in Graph Theory" (J. Adian Bondy and U. S. R. Murty, Eds.), pp. 399–406, Academic Press, San Diego/Toronto, 1984]. © 1989 Academic Press, Inc.

1. INTRODUCTION

In this paper we consider finite, undirected graphs which may have loops and multiple edges. Generally we use the terminology of F. Harary [3]. A graph H is a minor of a graph G if H is a subgraph of G or can be obtained from a subgraph of G by edge-contractions.

In their large series of fundamental papers on graph minors N. Robertson and P. D. Seymour [6–15] proved several interesting results on properties of graphs excluding a given planar graph as minor. They made several successful attempts to give partial answers to the following conjecture of K. Wagner (unpublished):

(1.1) (*Conjecture*[1]) *If C is a set of graphs such that no member of C is isomorphic to a minor of another, then C must be finite.*

The most outstanding question on the way to answer this problem is to give a characterization of the structure of those graphs not containing an arbitrary fixed graph as a minor. Robertson and Seymour made a corresponding conjecture in [15]. To give it here we need some definitions first.

A graph G is the *clique-sum* of graphs G_1 and G_2 if it can be obtained from G_1 and G_2 by choosing a clique from each (of the same size), deleting

[1] See the Acknowledgments at the end of this paper.

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the edges of the cliques, and pairwise identifying the vertices of one clique with those of the other. Let us write $G = cs(G_1, G_2)$ in this case.

The *completion* $C(K)$ of a class K of graphs is the smallest set including K, closed under clique-sum. A *k-vertex extension* of a graph G is a graph H such that there exist k vertices of H the deletion of which yield G.

A surface is a compact 2-manifold (without boundary). In this paper we are only concerned with connected surfaces. N. Robertson and P. D. Seymour [15, Conjecture 5.1] conjectured[1]:

> For any graph H there is a number k and a surface S such that every graph not containing H as a minor is in the completion of k-vertex extensions of graphs embeddable in S.

A consequence of results (2.4) and (3.1) of this article is a disproof of this conjecture.

2. Minors of T-Grids

The main result of N. Robertson and P. D. Seymour in [10] is the following theorem:

(2.1) *For every planar graph H, there is a number w such that every graph with no minor isomorphic to H has tree-width $\leqslant w$.*

The definition of tree-width can be found in [8]. Key structures in the proof of (2.1) are the *n-grids* ($= n \times n$*-grids*) Q_{nn} defined (more generally) as follows: For integers n, $m \geqslant 2$, an $n \times m$-grid Q_{nm} is a simple graph with vertex set $\{q_{ij}\colon 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m\}$, in which q_{ij} and $q_{i'j'}$ are adjacent if $|i - i'| + |j - j'| = 1$. The following fact is easy to see (cf. [10, 12]):

(2.2) *For every planar graph H there is an integer n such that H is isomorphic to a minor of the n-grid Q_{nn}.*

A graph arising from three disjoint grids Q_{nq}, R_{nr}, S_{ns} with vertex sets $\{q_{ij}\}$, $\{r_{ij}\}$, $\{s_{ij}\}$, respectively, identifying their vertices q_{i1}, r_{i1}, s_{i1} to a vertex p_i for every $i = 1, 2, \ldots, n$ and unifying multiple edges, is called a T-grid $T_n(q, r, s)$. The path induced by the vertices p_i is denoted by P, and C^k is the subgraph of $T_n(q, r, s)$ induced by the vertices q_{kj}, $j = 1, 2, \ldots, q$, r_{kj}, $j = 1, 2, \ldots, r$, and s_{kj}, $j = 1, 2, \ldots, s$.

In Figs. 1a and 1b two representations of $T_{10}(6, 2, 2)$ are given; the dotted and the fat lines indicate P and C^3, respectively. Every finite graph

[1] See the acknowledgments at the end of this paper.

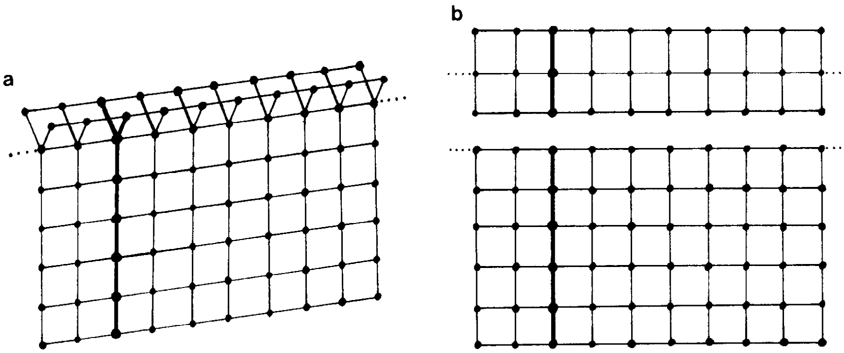

FIGURE 1

can be represented in the Euclidean plane in such a way that all crossings lie on a straight line (cf., e.g., R. K. Guy [2]). So by constructing "bridges" there it is straightforward to prove:

(2.3) *For every graph H there is an integer n such that H is isomorphic to a minor of the T-grid $T_n(n, n, 2)$.*

But this is not true after restricting to T-grids $T_n(n, c, c')$ with constant c and c'. We shall show this in case $c = c' = 2$; a general proof proceeds analogously. Abbreviate the T-grid $T_n(n, 2, 2)$ by Y_n and its "subgrids" Q_{nn}, R_{n2}, S_{n2} by Q, R, S, respectively.

(2.4) *There is an integer k such that the complete graph K_k is not isomorphic to a minor of any Y_n.*

Proof. Assume to the contrary that for every k there is an integer $n = n(k)$ such that K_k is isomorphic to a minor of Y_n. Moreover it can be assumed without loss of generality that all vertices of Y_n are marked with numbers 1, 2, ..., k in such a way tat K_k results from Y_n by contracting all edges between equally marked vertices.

Then the vertex set of Y_n is partitioned into k subsets inducing connected subgraphs ("countries") $A_1, A_2, ..., A_k$, respectively, of Y_n, which are pairwise neighbouring (by some further edges of Y_n).

So there are at most four countries which avoid P. Hence for every t with $2t-1 \leqslant k-4$ there are $2t-1$ countries $A_1, ..., A_{2t-1}$ which have representative vertices $\mathbf{1}, \mathbf{2}, ..., \mathbf{2t-1}$, respectively, on P, following each other in this order there.

Since A_i and A_{t+i} $(1 \leqslant i \leqslant t-1)$ are neighbouring, $\mathbf{i}$ and $\mathbf{t+i}$ are joined by a path P_i in Y_n which avoids all other countries. So the set of all P_i contains $t-1$ mutually disjoint paths joining the corresponding representative vertices.

Every path P_i crosses $C^{t'}$ (where t' is the index of the vertex $P_{t'} = \mathbf{t}$ on P) in a vertex $v_i \neq \mathbf{t}$; with the exception of at most two (because $c = c' = 2$) all these vertices v_i lie on Q. Consider three of them: v^{-1}, v^0, v^1, following each other in this order on (the path) $C^{t'} \cap (Q \backslash P)$, and those (unique) subpaths S^{-1}, S^0, S^1 of the three paths P_i with v^{-1}, v^0, v^1, respectively, containing these vertices and exactly two vertices of P, their end vertices. Notice that S^{-1}, S^0, S^1 are mutually disjoint.

Because of our assumption, v^{-1} and v^1 are joined by a path containing only vertices of their countries. Since in Q v^{-1} and v^1 are separated by S^0, this path must cross $(C^c \cup C^d) \cap (R \cup S \backslash P)$, where c, d are the indices of the endvertices of S_0 (on P).

$(C^c \cup C^d) \cap (R \cup S \backslash P)$ has only four vertices. So a set $\{v^{-5}, v^{-4}, ..., v^5\}$ defined as $\{v^{-1}, v^0, v^1\}$ above and used like that yields a contradiction.

Our conclusion shows that such a set exists if $l \geqslant 12$ or $k \geqslant 27$, respectively. This is improved to the best possible one ($k = 8$) in Section 4.

3. Clique-Sums and T-Grids

Let S be a fixed surface. Moreover denote by the same S the class of all graphs embeddable in S, which will not lead to confusion in the following. For a natural number k let S^k denote the class of all k-vertex extensions of graphs embeddable in S, and abbreviate $T_n(n, 2, 2)$ by Y_n again.

(3.1) *For every surface S and every natural number k there is a natural number n such that $Y_n \notin C(S^k)$.*

Proof. If G is a graph, (X, Y, Z) is a partition of $V(G)$, no vertex in X is adjacent to any vertex in Z, and every two members of Y are adjacent, let us say that G is made by overlapping G_1 and G_2, where $G_1 = G|(X \cup Y)$ and $G_2 = G|(Y \cup Z)$, and $G|X$ means the restriction of G to X. We denote this by $G = G_1 + G_2$. If C is a class of graphs, let $\langle C \rangle$ be the class constructible from members of C by repeated overlapping. For the proof of (3.1) it suffices obviously to prove:

(3.2) *For a fixed surface S and $k \geqslant 0$ there exists $n \geqslant 0$ such that no member of $\langle S^k \rangle$ has a minor isomorphic to Y_n.*

A drawing of a graph G in a surface is *simplicial* (say) if $G \neq K_4$ and every triangle of G (= circuit of three edges) bounds a region. For $g \geqslant 0$, let D_g be the class of all graphs which can be drawn in some (connected) surface of genus $\leqslant g$, and let E_g be the class with simplicial drawings in such a surface. Let D_g^k be the class of $(\leqslant k)$-vertex extensions of members of D_g, and define E_g similarly.

(3.3) *For* $g \geqslant 0$, $D_g \subseteq \langle E_g^{3g+1} \rangle$.

This is proved by induction on g. Let $G \in D_g$. If $G = K_4$ the claim holds, and so we may assume that there is a triangle T of G not bounding a region. Then either

(i) T is a separating triangle, and $G = G_1 + G_2$ where G_1, $G_2 \in C_g$ are smaller than G or

(ii) T is not a separating triangle, and its drawing is a nonnull-homotopic closed curve, and $G \backslash V(T) \in D_{g-1}$, where the result follows by induction.

From (3.2) and (3.3), it suffices to prove:

(3.4) *For* g, $k \geqslant 0$ *there exists* $n \geqslant 0$ *such that no member of* $\langle E_g^k \rangle$ *has a minor isomorphic to* Y_n.

To see this remember that any member of D_g^k is a subgraph of a member of $\langle E_g^{k+3g+1} \rangle$.

(3.5) *For* $g \geqslant 0$ *and* n *sufficiently large, no member of* D_g *has a minor isomorphic to* Y_n.

This is clear, since the genus of Y_n is growing with n.

If H is a minor of G and H is isomorphic to Y_n (briefly, H is a Y_n-minor of G), by a row of H we mean the set of edges of H forming one of the "horizontal" paths of length $n-1$ in Q_{nn}.

(3.6) *Let* H *be a* Y_n*-minor of* G, *where* $G = G_1 + G_2$ *and* $|V(G_1 \cap G_2)| < n$. *Then exactly one of* $E(G_1)$, $E(G_2)$ *includes a row of* H.

For not both $E(G_1)$, $E(G_2)$ can include such a row, since there are n mutually disjoit paths of G between any two rows. On the other hand, each row is a subset of $E(P)$ for some path P of G, and these paths are mutually disjoint; and at least one of these n paths is disjoint from $V(G_1 \cap G_2)$, since $|V(G_1 \cap G_2)| < n$. The claim follows.

(3.7) *Let* H *be a* Y_n*-minor of* G, *where* $G = G_1 + G_2$ *and* $|V(G_1 \cap G_2)| \leqslant 3 < n$. *Let* $E(G_2)$ *include no row of* H. *Then there is a subgraph* G_2' *of* G_2 *with* $G_1 \cap G_2 \subseteq G_2'$, *such that* H *is a minor of* $G_1 + G_2'$, *and* G_2' *is planar and can be drawn in a disk with* $V(G_1 \cap G_2)$ *on the boundary.*

The proof is easy, by examining all separations of Y_n of order $\leqslant 3$.

(3.8) *For all g, $k \geqslant 0$ there exists $n \geqslant 0$ with the following property. Let $G = G_0 + G_1 + \cdots + G_t$, where for $1 \leqslant i < j \leqslant t$, $G_i \cap G_j \subseteq G_0$, and let $G_0 \in E_g^k$. For every Y_n-minor H of G, one of $E(G_1)$, ..., $E(G_t)$ includes a row of H.*

First we prove this for $k = 0$. Then, since G_0 is simplicial it follows that each $V(G_i \cap G_0)$ has $\leqslant 3$ members and lies in the boundary of some region of the drawing of G_0. For $1 \leqslant i \leqslant t$, if $E(G_i)$ includes no row of H choose $G_i' \subseteq G_i$ such that $G_0 \cap G_i \subseteq G_i'$ as in (3.7); then $G_0 + G_1' + \cdots + G_t' \in C_g$, and H is a minor of it, contrary to (3.5) for n sufficiently large. Now for the general case $k \geqslant 0$. Choose $m > k + 3$ so that (3.8) is satisfied with $k = 0$ and $n = m$; and choose n such that if $X \subseteq V(Y_n)$ and $|X| \leqslant k$ then $Y_n \backslash X$ has a subgraph isomorphic to Y_m. Let $G_0, ..., G_t$ be as in (3.8). Choose $X \subseteq V(G_0)$ with $|X| \leqslant k$ and $G_0 \backslash X \in E_g$. Then $G \backslash X$ has a Y_m-minor $H' \subseteq H$, where every row of H' is a subset of a row of H. From the choice of m, there exists i with $1 \leqslant i \leqslant t$ such that $E(G_i \backslash X)$ includes a row of H'. Let $i = t$ say. Since $m > k + 3$, $G_0 + G_1 + \cdots + G_{t-1}$ includes no row of H (arguing as in (3.6), for there are m disjoint paths of G between any row of H and any row of H'). By (3.6), G_t includes a row of H, as required.

Now (3.4) may be deduced from (3.8) as follows. Given g, k let $G \in \langle E_g^k \rangle$. Then there is a tree T and for each $t \in V(T)$ an induced subgraph G_t of G which is in E_g^k, such that

(i) $U(G_t : t \in V(T)) = G$;

(ii) for t, $t' \in V(T)$, adjacent in T, every two members of $V(G_t \cap G_{t'})$ are adjacent in G;

(iii) for t, t', $t'' \in V(T)$, if t' lies on the path of T between t and t'' then $G_t \cap G_{t''} \subseteq G_{t'}$.

For each $e \in E(T)$ with ends t_1, t_2, let T_1, T_2 be the two components of $T \backslash e$, where $t_i \in V(T_i)$ $(i = 1, 2)$. Orient e from t_1 to t_2 if $U(E(G_t : t \in V(T_2)))$ includes a row of H (where H is a Y_n-minor of G). Thus every edge of G receives a unique orientation, by (3.6), since every $|V(G_t \cap G_{t'})| \leqslant k + 3$ for t, $t' \in V(T)$, adjacent (if $n > k + 3$). Since $|V(T)| > |E(T)|$, there exists $t_0 \in V(T)$ which is not the tail of any edge of T. But that contradicts (3.8) for n sufficiently large and completes the proof of (3.1).

4. Minors K_k of Y_n

(4.1) *The complete graph K_k is isomorphic to a minor of some Y_n if and only if $k \leqslant 7$.*

Proof. Figure 2 shows that K_7 is a minor of $T_{14}(6, 2, 2)$, and Y_{14}, respectively. It must be read as Fig. 1b. So assume again as in the proof of

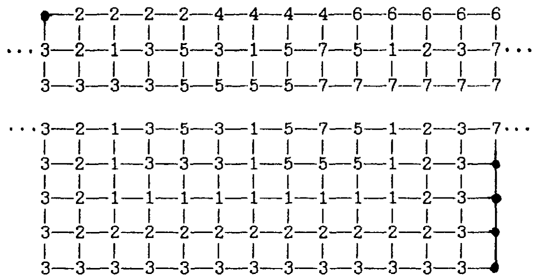

FIGURE 2

(2.4) that there is an integer n such that Y_n can be partitioned into eight pairwise neighbouring "countries" $A_1, A_2, ..., A_8$ whose vertices are marked by 1, 2, ..., 8, respectively. Abbreviate again the three grids Q_{nn}, R_{n2}, S_{n2} constituting $Y_n = T_n(n, 2, 2)$ by Q, R, S, respectively, and remember that the path P is the unique maximal common subgraph of Q, R, S (regarded as parts of Y_n). Assume first that there is a country—marked by 8—containing no vertex of $R \cup S$. Then we can choose a set of eight representative vertices **1**, **2**, ..., **8** of these countries and seven paths **8/j**, whose vertices are marked by 8 or j, joining **8** and **j**, $j = 1, 2, ..., 7$, completely lying on Q. Let these paths by cyclically ordered around the (connected) 8-part of Y_n, as is indicated up to cyclic permutation in Fig. 3, and let F be any family of $\binom{8}{2}$ paths **i/j**, whose vertices are marked by i or j, joining the representatives **i** and **j**, which contains the above paths **8/j**.

If the path **1/5** of F does not leave Q, then **2/6** canot be realized. So **1/5** contains at least one vertex of $R \cup S \setminus P$. Let **1**/5 and 1/**5** be the first vertices of this kind arrived from **1** and **5**, respectively. Because of the grid structure of Y_n **1**/5 $\neq$ 1/**5**; they can belong to the same or to different ones of R, S.

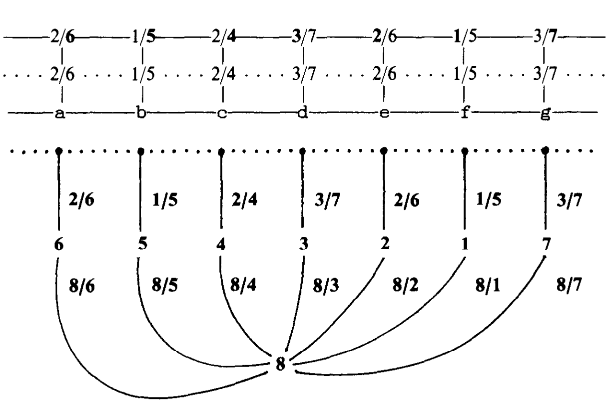

FIGURE 3

By definition their neighbours on P also belong to **1/5**. The fact that a vertex v belongs to **i/j** is denoted by $v = i/j$ in the following, whereas $v = i$ means that v has the mark i; i/j in Fig. 3 is defined correspondingly. The course of the final parts of **1/5** drawn in Fig. 3 demonstrates the partition of Q by **1/5** and the 8-part of Y_n. These final parts can contain more than one vertex of P.

Proceeding analogously with **2/6** and **3/7** we get a unique (up to cyclic permutation) ordering of **1**/5, 1/**5**, **2**/6, 2/**6**, **3**/7, 3/**7** along P as given in Fig. 3. Moreover there 2/**4** is inserted in its unique place. But notice again that these vertices can be distributed on R, S in any way.

The labels a, b, ..., g in Fig. 3 are assigned to those neighbouring vertices of the i/j's, which belong to $R \cup S \backslash P$, but are different from **i**/j, i/**j**.

Let p be a path (on Y_n) containing a vertex v on P and q a path on $R \cup S$ joining two vertices of P which are separated by v there and having no other vertices of P. Then q is said to bridge p in the (unique) neighbouring vertex r of v belonging to q. Moreover, every path q' having q as a subgraph is said to bridge p in r, too.

Let **1/5** be fixed. Then one can see (cf. Fig. 3) that **2/6** must bridge **1/5** in b or in f, i.e., $b = 2/6$ or $f = 2/6$. Similarly **3/7** must bridge **1/5** in b or in f, i.e., $b = 3/7$ or $f = 3/7$. If $b = 3/7$, $f = 2/6$, then **4/6** cannot bridge **1/5** in b. So **4/6** must bridge **3/7** in d, **2/6** in e, **1/5** in f, and **3/7** in g (cf. Fig. 3), and it follows that $d = 4/6$, $g = 4/6$. But then **1/5** cannot bridge **3/7** in d or g and so cannot be realized.

Thus, $b = 2/6$, $f = 3/7$.

1/5 and **2/6** mus bridge **3/7** in d and in g. If $d = 1/5$, $g = 2/6$, then **2/4** cannot bridge **3/7** in d which implies $f = 2/4$ contradicting the above statement.

Thus, $d = 2/6$, $g = 1/5$.

1/5 and **3/7** bridge **2/6** in a and in e. If $e = 1/5$, then **3/7** cannot bridge **2/6** in e which implies $b = 3/7$ contradicting the above statement.

Thus, $a = 1/5$, $e = 3/7$.

1/4 cannot bridge **3/7** in $d = 2/6$. Hence **1/4**, coming from **4**, first must cross **1/5** (in a vertex labeled by 1) and then must bridge **2/6** in a, i.e., $a = 1/4$, which together with $a = 1/5$ yields $a = 1$. But then **3/5** and **5/7** cannot bridge **2/6** in a which implies that they both must bridge **2/4** in c, i.e., $c = 3/5$ and $c = 5/7$, respectively, and so $c = 5$.

4/7 cannot bridge **1/5** in $b = 2/6$. Hence **4/7** must bridge **2/6** in e, i.e., $e = 4/7$, which together with $e = 3/7$ yields $e = 7$. But then **1/3** cannot bridge **2/6** in e which implies that it must bridge **2/4** in c contradicting $c = 5$.

So we have shown:

(4.2) *Each of the countries into which Y_n is partitioned contains at least one vertex of $R \cup S$.*

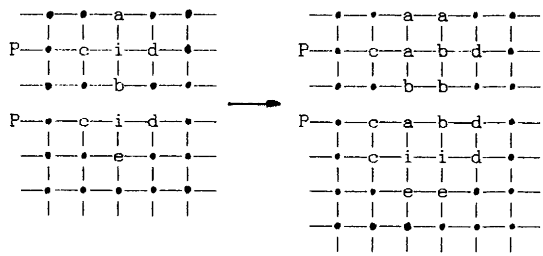

FIGURE 4

If there is a vertex on P labeled by i, and neither of its neighbours in $R \cup S \backslash P$ have label i, then the label i on P can be omitted at this place by slightly enlarging Y_n (i.e., using a Y_m with $m > n$) and relabeling the vertices concerned as sketched in Fig. 4.

So we can assume without loss of generality:

(4.3) *Every vertex on P has a neighbour on $R \cup S \backslash P$ with the same label.*

If there is a country A_i one of whose maximal Q-parts partitions the residual Q-part of Y_n into two nonempty regions (cf. Fig. 5), then because of (4.3) there are only two possibilities of realizing connections between parts of other countries in different regions; i.e., one of these regions has only vertices of at most two countries a and b. As sketched in Fig. 5, these vertices can be relabeled such that there is at most one vertex not labeled by i in the region concerned ("suck out" these labels along corresponding paths). But exchanging the labelings of $R \backslash P$ and $S \backslash P$ on the right of the a's in the middle of Fig. 5 and relabeling two a's shows that we can assume without loss of generality:

(4.4) *No maximal Q-part of some country partitions the Q-part of Y_n.*

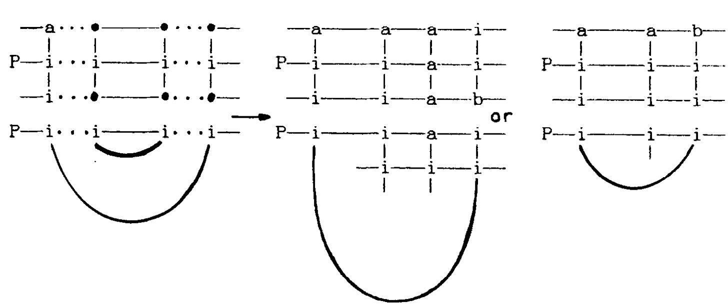

FIGURE 5

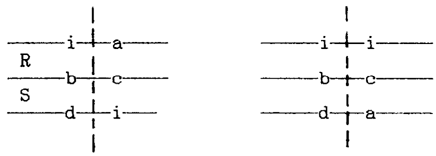

FIGURE 6

Then every two maximal Q-parts of some country are joined by a path on $R \cup S$ belonging to the same country.

Moreover (4.4) implies:

(4.5) *The vertices of $R \cup S$ labeled by some mark i induce a connected subgraph G_i.*

If there are two neighbouring vertices b, c on P which are neighbouring with vertices b', c', respectively, of G_i, then we can arrange by exchanging certain labelings of $R\backslash P$ and $S\backslash P$ that both b' and c' belong to R (or, analogously, to S). The nontrivial case is illustrated in Fig. 6: exchange the labels on the left or on the right of the dotted line there. By (4.5) $a = i$ or $d = i$ or $b = c = i$. In every case all neighbourhoods between the countries labeled by a, b, c, d, i are preserved under the considered exchange.

By successive relabeling in this way along P (from left to right) we can arrrange that:

(4.6) *For every mark i the graph G_i of* (4.5) *is induced by the vertices of a path p on $R\backslash P$ or on $S\backslash P$ and some further vertices on P neighbouring with p and on $S\backslash P$ or $R\backslash P$, respectively, neighbouring with such vertices on P.*

Figure 7 shows how we can cancel labels i lying on that one of $R\backslash P$, $S\backslash P$ which does not contain the path p of (4.6), without destroying neighbourhoods of countries. Eventually y_n must be enlarged (left case of Fig. 7).

In this way we can arrange that:

(4.7) *For every mark i the country A_i avoids $R\backslash P$ or $S\backslash P$.*

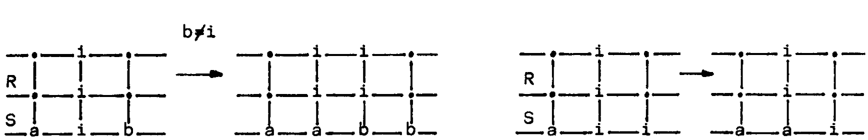

FIGURE 7

By (4.2), (4.3), and (4.5) each of our countries is represented on $R \cup S \backslash P$ and its subgraph on $R \cup S$ is connected; by (4.7) this graph lies on R or on S. Since there are eight countries, there are (at least) four countries A_1, A_2, A_3, A_4 (say) avoiding $R \backslash P$ (or $S \backslash P$). If A_1, A_2, A_3 are mutually neighbouring on S, then one of them has a vertex on P contradicting (4.3). By the same argument it is impossible, too, that one country is neighbouring with three others in S. So let two countries of A_1, A_2, A_3, A_4 be neighbouring on S if and only if they follow each other in this sequence. But then A_1 and A_3 as well as A_2 and A_4 must be neighbouring on Q which is impossible.

This completes the proof of (4.1).

Acknowledgments

After this article was submitted we received N. Robertson and P. D. Seymour's preprints [20–23] and obtained information on [24]. In [24] a positive solution of K. Wagner's conjecture (1.1) by N. Robertson and P. D. Seymour was announced. Moreover in [23] N. Robertson and P. D. Seymour mention (at the top of page 3) that their Conjecture 5.1 from [15] is false (without proof), and state and prove a correct version of it. We are grateful to N. Robertson and P. D. Seymour for sending us their stimulating articles. Moreover we thank the referee for several helpful comments and suggestions. In particular we are grateful to him for permitting us to use his proof of (3.1), which is much simpler than our proof.

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